An imperfect repair model based on reduction of virtual age and uniform distributed degrees of repair

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Abstract

In this paper we discussed the parameter estimation of a general repair model using a Weibull intensity. We consider an incomplete repair model with the impact of repair after failure is not minimal as in the non-homogeneous Poisson process and not "as good as new" as in renewal process but lies between these boundary cases. Moreover repairs affect the failure intensity at any instant via a virtual age process. When the system fails, a repair is allowed performed with a uniform distributed degree of repair including as special cases perfect, minimal and imperfect repair models. The maximum likelihood estimator is considered for determining the estimations of the model parameters. Simultaneous confidence regions based on the likelihood ratio statistics are developed for the parameter estimators of the Weibull intensity. The obtained results are applied on sets of simulated data. Furthermore, a well known data on airplane air-conditioning failure data (Plane 7914) have been analyzed.

Keywords: Reliability; Weibull intensity; Imperfect maintenance; ML estimation; Virtual age.

1. Introduction

In this research we discussed the parameter estimation of a general repair model introduced by Last and Szekli [1] using a Weibull intensity. The most commonly used models for the failure process of a repairable system are known as minimal repair or as bad old and perfect repair or as good as new. It is well known in practice that the reality is between these two boundary cases. The repair may not yield a functioning system which is as good as new and the minimal repair assumption seems to be too pessimistic in repair strategies. Therefore, it is seen that the imperfect repair is of great significance in practice. The repair effect in this paper is expressed by a reduction of the system virtual age introduced by Kijima et al. [2] and Kijima [3].

Here we can note that most of the models concerning the modeling of repairable systems identify the minimal repair and the imperfect repair actions. Naturally, this popular assumption is a very unreal one. The model introduced in this paper is more flexible than much other models. It includes the special cases of classical perfect repair, minimal repair, imperfect repair and general repairable system of Kijima [3]. When the system fails, a repair is allowed performed with a uniform distributed degree of repair. This assumption gives more flexibility in order to model repairable systems.

Recently, statistical studies of parameter estimation in systems with various degrees of repair using virtual age models are given by Bathe and Franz [4], Stadje and Zuckerman [5], Gasmi et al. [6], Baxter et al. [7], Doyen and Gaudoin [8] and Gasmi [9-11].

This failure intensity used in this paper is a Weibull intensity which is especially used as a failure model analyzing the reliability of different types of systems and can characterize the probabilistic behavior of a large number of real phenomena.

Estimations of the unknown parameters included in the failure intensity and used in the general repair model is an interesting problem in reliability analysis. The main objective of this paper is the development of confidence estimations for the parameters of the general repair model in the case of a uniform distributed degree of repair.

We will particularly use the Kijima's type 2 imperfect repair model to describe the age reduction after repair actions.

2. The Kijima type 2 repair model

Now, we consider a Kijima type 2 repair model, based on the reduction of virtual age, introduced by Last and Szekli [1]. The system starts working with an initial prescribed failure rate $\lambda_1(t) = \lambda(t)$. Let t_1 denote the random time, where the system falls out. At this time t_1 the system will be repaired with the random degree of repair z_1. We note that this degree is always between 0 and 1 where the case of 0 corresponds with the perfect repair and the case with 1 corresponds with the minimal repair. The age of the system is decreased to $z_1 \cdot t_1$ which is called the virtual age of the system at time t_1 and is denoted by v_1. The distribution of the time until the next failure has then the failure intensity $\lambda_2(t) = \lambda(t - t_1 + v_1),\ t_1 \le t < t_2$. We assume now that for $k \ge 1$, t_k is the time of the k-th failure and that z_k is the degree of repair at that time.

After repair the failure intensity of the $(k+1)$-th waiting time until the next failure is determined by $\lambda_{k+1}(t) = \lambda(t - t_k + v_k),\ t_k \le t < t_{k+1},\ k \ge 0$. Where v_k is from Kijima's type 2 imperfect repair model $v_k = z_k\left(v_{k-1} + t_k - t_{k-1}\right), k \ge 1$, and $v_0 = 0$, that is the repair resets the intensity of failure proportional to the virtual age.

We note that the process defined by $v(t) = t - t_k + v_k,\ t_k \le t < t_{k+1},\ k \ge 1$, is called the virtual age process [9,11].

Throughout this paper we suppose that all repair times are small and can be neglected moreover repairs affect the failure intensity at any instant via a virtual age process from type Kijima 2. We assume that after failure, one of the three following cases is possible, a perfect repair, a minimal repair or an imperfect repair with uniform distributed degree of repair.

In this paper, we assume that the baseline failure intensity of the system is from Weibull type $\lambda(t,\alpha,\beta) = \frac{\beta}{\alpha^{\beta}} x^{\beta-1}, \alpha > 0, \beta > 0$. This intensity is an extremely important intensity to characterize the probabilistic behavior of a large number of real phenomena and is therefore used as a failure model in analyzing the reliability of many types of real data. Our purpose is to estimate the two parameters α and β.

3. Parameter Estimation

We consider now a marked point process $\Phi = ((t_k, z_k))$[1]. Φ is described by the counting process $\{N(t),\ t \geq 0\}$ and the corresponding intensities $\lambda_t = \lambda(v(t),\alpha,\beta)$.

We note that:

- If the degree of repair $z_k = 0$, for all $k = 1,\ldots,N(t)$, only perfect repairs appears in the model, and we obtain a Renewal Process (RP), denoted by model 2.
- If the degree of repair $z_k = 1$ for all $k = 1,\ldots,N(t)$, only minimal repairs appears in the model, and we obtain a non-homogeneous Poisson Process (NHPP), denoted by model 3.
- If the degree of repair $z_k \in (0,1)$ we have then an imperfect repair (IR) (model 1).

The loglikelihood function for observation of point processes is of the form (Liptser and Shiryayev) [12]

$$\ln L(t,\alpha,\beta) = \sum_{k=1}^{N(t)} \ln\left(\lambda(v_{k-1} + t_k - t_{k-1})\right) - \int_0^t \lambda\big(v(s)\big)ds. \tag{1}$$

After substitution of the Weibull failure intensity, we get then the following LL function:

$$\ln L(t,\alpha,\beta) = (\beta - 1) \sum_{k=1}^{N(t)} \ln(v_{k-1} + t_k - t_{k-1}) + N(t)(\ln\beta - \beta\ln\alpha) - \frac{1}{\alpha^{\beta}} S_1, \tag{2}$$

where $S_1 = \sum_{k=1}^{N(t)} \{(v_{k-1} + t_k - t_{k-1})^{\beta} - {v_k}^{\beta}\} + (t - t_{N(t)} + v_{N(t)})^{\beta}$ and $N(t)$ denotes the number of failures until t.

Using the standard maximum likelihood approach to maximize equation (2) we can see that it is possible to explicitly determine the scale parameter α and we obtain then the usual results for the Power-Law Process:

$$\hat{\alpha} = \left({S_1}/{N(t)} \right)^{1/\beta} \tag{3}$$

We remark that by setting (3) in the equation (2) some analytical simplifications of the structure are possible. The remaining loglikelihood function is then:

$$\ln L(t, \beta) = (\beta - 1) \sum_{k=1}^{N(t)} \ln(v_{k-1} + t_k - t_{k-1}) + N(t)(\ln \beta - 1 - \ln S_1) \tag{4}$$

The estimation of the shape parameter β can be found by the numerical solve of the following equation:

$$\frac{1}{\hat{\beta}} + \frac{1}{N(t)} \sum_{k=1}^{N(t)} \ln(v_{k-1} + t_k - t_{k-1}) - \frac{S_2}{S_1} = 0, \tag{5}$$

where

$$S_2 = \sum_{k=1}^{N(t)} \left\{ (v_{k-1} + t_k - t_{k-1})^{\beta} \ln(v_{k-1} + t_k - t_{k-1}) - {v_k}^{\beta} \ln(v_k) \right\} + (t - t_{N(t)} + v_{N(t)})^{\beta} \ln\left(t - t_{N(t)} + v_{N(t)}\right). \tag{6}$$

4. Simultaneous confidence regions

The most efficient way to calculate simultaneous confidence regions for two or more parameters is based on the likelihood ratio. If the parameters of the lifetime distribution are estimated with the maximum likelihood method, then it is possible to calculate for instance the reliability of the product. It is well known (see Barndorff-Nielsen and Blaesild [13]), that in general the loglikelihood ratio

$$LQ = 2\left\{ \ln L\left(t, \hat{\alpha}, \hat{\beta}\right) - \ln L\left(t, \alpha, \beta\right) \right\}$$

is asymptotically χ^2- distributed with 2 degrees of freedom, where $\hat{\alpha}$, $\hat{\beta}$ are the maximum likelihood estimations of the parameters of interest α and β. This fact can be used to determine simultaneous confidence regions.

The simultaneous confidence region is defined by the inequality $LQ \leq \chi^2{}_{1-\mu,2}$, where $\chi^2{}_{1-\mu,2} = -2\ln\mu$ is the (1-μ)- quantile of the χ^2- distribution with 2 degrees of freedom.

Now, we observe n independent failure repair processes. Let $r \in \{1,2,\ldots,n\}$, we introduce the following notations:

- $N^r(t)$ - the number of failures until t for the $r-$th failure repair process,
- $t_{r,1}, \ldots, t_{r,N^r(t)}$ - failure times of the $r-$th failure repair process,
- $v_{r,1}, \ldots, v_{r,N^r(t)}$- the virtual ages of failures until t for the $r-$th failure repair process,
- $L_r\,(t,\alpha,\beta)$ - the likelihood function of the $r-$th failure repair process.

The loglikelihood function of the r-th failure repair process is defined as before in equation (2). We get the following LL function:

$$\begin{aligned}\ln L_r(t,\alpha,\beta) = (\beta-1)\sum_{k=1}^{N^r(t)} \ln(v_{r,k-1} + t_{r,k} - t_{r,k-1}) \\ +N^r(t)(\ln\beta - \beta\ln\alpha) - \frac{1}{\alpha^\beta} S_{r,1},\end{aligned} \tag{7}$$

Where $S_{r,1} = \sum_{k=1}^{N^r(t)} \left\{ \left(v_{r,k-1} + t_{r,k} - t_{r,k-1}\right)^\beta - v_{r,k}{}^\beta \right\} + (t - t_{N^r(t)} + v_{r,N^r(t)})^\beta$.

By using the loglikelihood ratio, we obtain then the simultaneous confidence region for n independent failure repair processes as follows:

$$\begin{aligned}\frac{1}{\alpha^\beta}\sum_{r=1}^{n} S_{r,1} + (\hat{\beta}-\beta)\sum_{r=1}^{n}\sum_{k=1}^{N^r(t)} \ln\left(v_{r,k-1} + t_{r,k} - t_{r,k-1}\right) \\ +\left(\ln\hat{\beta} - \ln\beta - 1 - \hat{\beta}\ln\hat{\alpha} + \beta\ln\alpha\right)\sum_{r=1}^{n} N^r(t) = -n\ln\mu\end{aligned} \tag{8}$$

5. Simulation study

In this section, we present numerical results based on a large simulation study. This study is considered to apply the previous theoretical results to simulated lifetime data and has been made by writing some computer programs with *Matlab 7*.

The simulation study to generate data of the IR process and following the Kijima type 2 repair model with uniform distributed degrees of repair has been done according to the following algorithm:

A. Initialize $k = 1$, $t_0 = 0$, $v_0 = 0$ and $d_k = 0$.

B. Generate a uniform distribution random variable u on the interval (0,1).

C. Evaluate $t_1 = -\alpha \ln^{\frac{1}{\beta}}(1-u)$.

D. Generate a uniform distribution random variable u on the interval $(0,1)$.

E. Solve the equation $(d_k + v_{k-1})^\beta - (v_{k-1})^\beta + \alpha^\beta \ln(1-u) = 0$.

F. Evaluate $t_k = t_{k-1} + d_k$.

G. If $t_k < t$, then $k \leftarrow k+1$ and go to step D.

H. Store the vector of failure times $T = (t_0, t_1, \ldots, t_{N(t)})$.

In the following example, let $\alpha = 1.2$ and $\beta = 3$. Table I illustrated two samples, the first is the uniform distributed degree of repair and the second is the corresponding failure times. We obtain $N(t) = 10$ failures until the time $t = 10$.

Table I: Failure times and repair degrees for $\alpha = 1.2$ and $\beta = 3$

z_k	0.4159	0.4900	0.3486	0.6980	0.1740	0.2247	0.8906	0.1236	0.1357	0.4553
t_k	1.3515	1.8118	2.4047	3.3637	3.4218	4.1991	4.6912	5.6913	7.9375	8.0719

In the following, parameter estimations of α and β are illustrated. A sample 1 was observed until $t = 100$. Let $\alpha = 1.2$, $\beta = 3$ and $r = 10$, where r is the number of simulations. The estimations of α, β and LL from data of sample 1 are given in Table II.

Table II: Estimations of α, β and LL from data of sample 1

$\hat{\alpha}$	$\hat{\beta}$	**LL**
1,1907	2,9978	-25,0579
1,2014	2,9880	-24,9123
1,2337	3,0314	-36,5588
1,2216	3,1723	-26,7767
1,2537	3,0977	-28,6813
1,2158	2,9855	-26,9009
1,1789	2,9456	-30,6481
1,1666	2,9134	-32,7947
1,1855	3,0615	-25,9890
1,2104	2,9219	-26,1210

The mean squared errors (MSE) of $\hat{\alpha}$ and $\hat{\beta}$ are given in Table III.

Table III: MSE of $\hat{\alpha}$ and $\hat{\beta}$

s	**50**	**100**	**500**	**1000**
MSE($\hat{\alpha}$)	0,0022	0,0018	0,0017	0,0015
MSE($\hat{\beta}$)	0,0993	0,0542	0,0514	0,0504

Based on the results booked on Table III, we could remark that if the number of simulations s increases, then the mean squared errors of $\hat{\alpha}$ and $\hat{\beta}$ decrease.

Simulations are carried out with different sets of parameters. From one set of parameters, the bias and variance of the estimators are estimated by their empirical version on 200 replicates. The estimations are given in Table IV.

Table IV: Estimations results considering an average of 200 simulations

	$\hat{\alpha}$	$\hat{\beta}$
Estimation	1,1903	2,9896
Empirical mean	1,2001	3,0370
Empirical variance	0,0016	0,0541

Figure 1 illustrates the LL function with respect to α and β in case of the model (first line in Table II). For fixed β equal 3 we obtain in Figure 2 the graph of the LL function with respect to α. By setting α equal 1.2, Figure 3 illustrated the the graph of the LL function with respect to β.

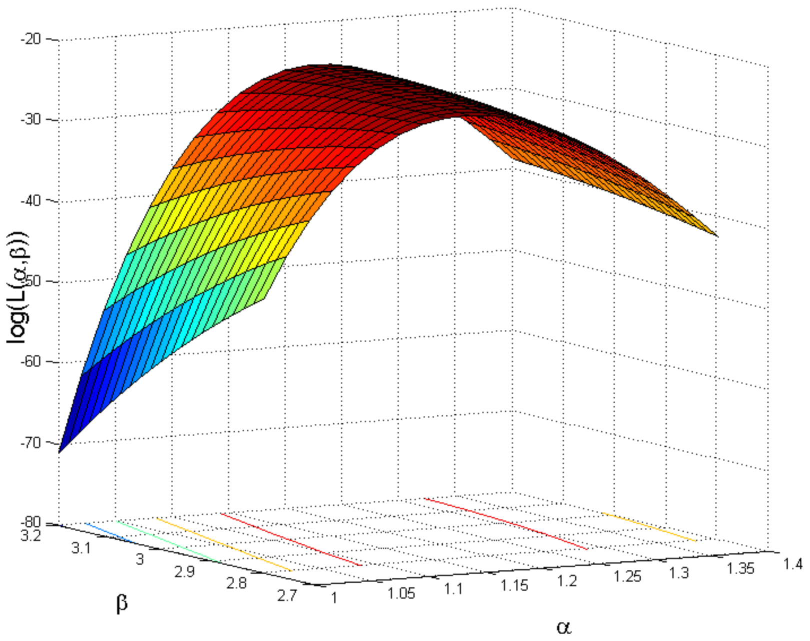

Figure 1. Graph of the LL function for one sample with respect to α and β

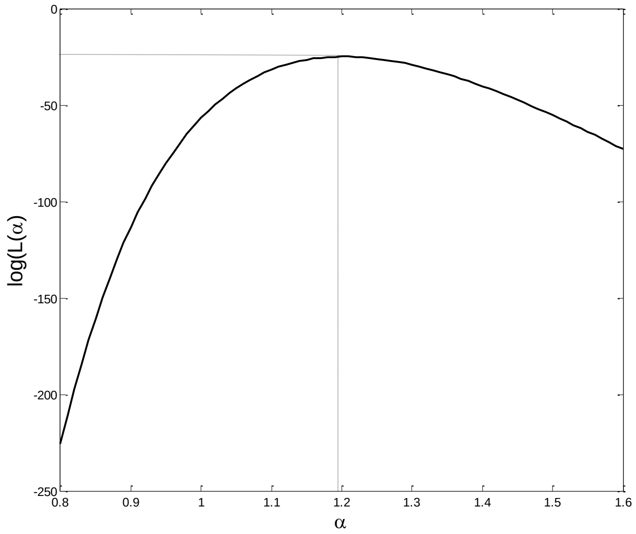

Figure 2. Graph of the LL function for one sample with respect to α

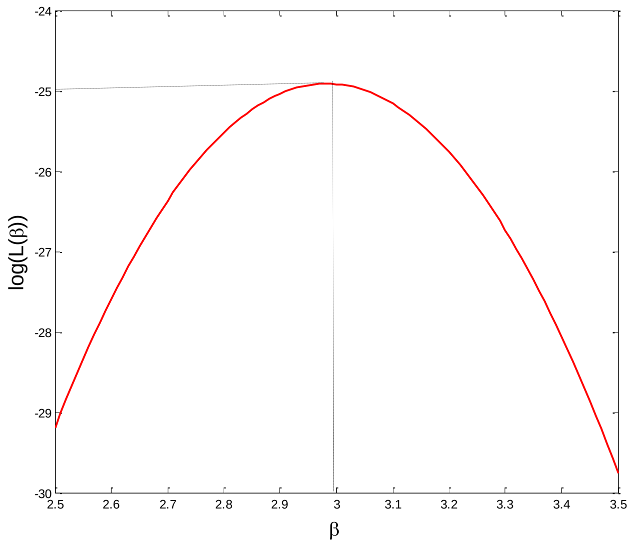

Figure 3. Graph of the LL function for one sample with respect to β

Based on the likelihood ratio statistic we obtain in Figure 4 the simultaneous confidence region of the parameter estimations $\hat{\alpha} = 1.2030$and $\hat{\beta} = 3.0375$ by given $\mu = 0.05$ for $n = 50$ (curve in line) and $n = 100$ (curve in dash).

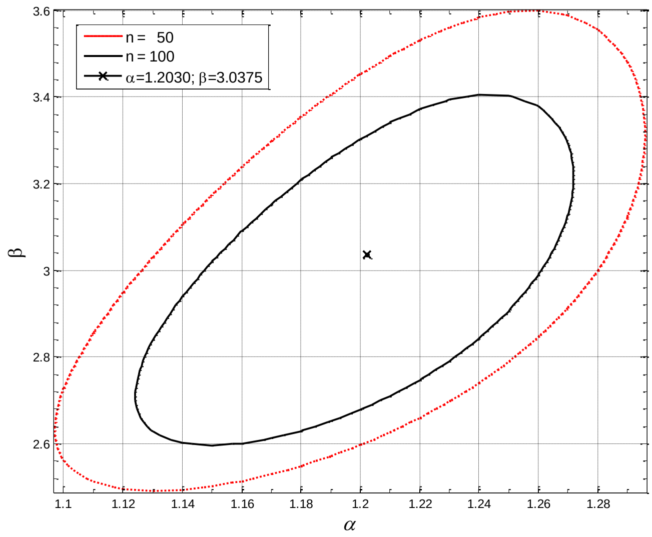

Figure 4. Simultaneous confidence region for the estimations of α=1.2 and β=3

Based on the previous analysis and on the results of Figure 4, we could remark that the simultaneous confidence region based on the likelihood ratio for $n = 100$ is smaller than that for $n = 50$. In the case of $n = 100$, the parameter α varies between 1.124 and 1.270 and the parameter β varies between 2.6 and 3.4.

6. Illustrative Example

We consider in this section a well known data on airplane air-conditioning failures on a fleet of Boeing aircraft (Plane 7914) given in Hollander and Wolfe [14]. For comparison purpose, we use the mean square of the difference between the empirical cdf and the fitted cdf, say MSD. We note that for the data from Plane 7914, the number of failures is $N(t) = 24$. The rest time after the last failure is assumed to be equal to zero. Our objective is to compare the model with uniform distributed degree of repair, denoted by model 1 and models using a fixed degree of repair. The models 2 and 3 are defined as before in Section 3. If the degree of repair $z_k = 0.5$ for all $k = 1, \ldots, N(t)$, we obtain the model 4. For comparison purpose, we use the mean square of the difference between the empirical cdf and the fitted cdf, say MSD. Note that MSD is computed by the relation:

$MSD = \frac{1}{N(t)} \sum_{k=1}^{N(t)} (\hat{F}_k - F_{E,k})^2$, where $\hat{F}_k$ and $F_{E,k}$ are the empirical and the estimated cdf computed at the cumulative failure times t_k .

Table V gives the inter-failure times for Plane 7914 and the uniform distributed degrees of repair used in model 1.

Table V: Inter-failure times from Plane 7914 and degrees of repair (model 1)

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
$\mathbf{x_i}$	50	44	102	72	22	39	3	15	197	188	79	88
z_k **model 1**	0.83	0.63	0.54	0.65	0.73	0.09	0.87	0.01	0.29	0.18	0.93	0.07
i	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
$\mathbf{x_i}$	46	5	5	36	22	139	210	97	30	23	13	14
z_k **model 1**	0.58	0.64	0.65	0.86	0.05	0.81	0.53	0.69	0.21	0.54	0.70	0.96

Table VI: Estimations of α, β, LL and MSD from data of Plane 7914

	$\hat{\alpha}$	$\hat{\beta}$	**MSD**	**LL**
Model 1	56.8875	0.9224	0,0010	-123.8080
Model 2	65,4085	1,0339	0,0014	-123,9939
Model 3	82,9261	1,0880	0,0057	-123,9455
Model 4	55,8537	0,9138	0,0011	-123,7917

The ML estimates of the parameters α and β, the MSD and the LL values were numerically evaluated for model 1, model 2, model 3 and model 4. The results are given in Table VI.

The empirical, the estimated cdf and the 95 % lower and upper confidence bounds for the cdf of the data from Plane 7914 are represented for all introduced models. The results are shown in figure [5].

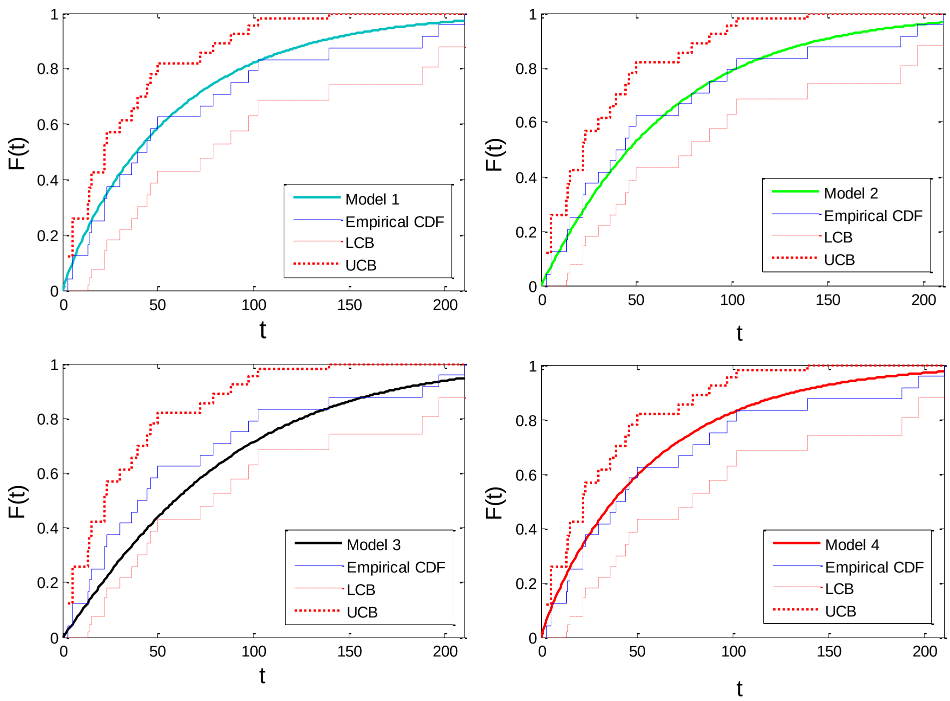

Figure 5. Cdf and empirical cdf of data from Plane 7914 model 1-4

Based on the results booked on Table 6 and on Figure 5, we conclude that model 4 fits the data better than model 2 and model 3 but model 1 with uniform distributed degrees of repair fits the data butter than model 4. We remark that the MSD for the model 1 ($MSD = 0.0010$) is smaller than the MSD for all other models.

7. Conclusions

In this paper we discussed the parameter estimation of a general repair model with various degrees of repair based on the virtual age principle of Kijima type 2 and using a Weibull intensity. The maximum likelihood estimators and the confidence regions based on the likelihood ratio statistics are obtained. The theoretical results presented in this paper have been applied on a set of simulated data. Furthermore, for illustrative purpose a well known and much discussed data on airplane air conditioning failures on a fleet of Boeing aircraft (Plane 7914) have been analyzed. The results obtained indicate that the model 1 with uniform distributed degrees of repair fits the data better than the model 2 (renewal process), model 3 (non-homogeneous Poisson process) and model 4 with fixed degree of repair equal to 0.5. Extensive simulations are required to illustrate the obtained theoretical results.

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