# Modeling model uncertainty in structural reliability: a variety of approaches

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**Abstract**

*For a long time, deterministic approaches have been taking account of model uncertainty through model coefficients. However, the model coefficient is not rigorously defined in the deterministic approach.*

*Following Ditlevsen's approach, this paper reminds that it is acceptable to propose a probabilistic representation of model uncertainty. A variety of probabilistic modelings for model uncertainty appear in the literature. Some common issues can be pointed out. All modelings include unknown parameters as well as a residual error term. These parameters, likely to be calibrated to improve the predictability of the model, are incorporated either to the base model, or, in the most common representation, to a corrective term representing the systematic bias of the base model. The paper presents these two main modelings, as well as a proposal of more general representation and an overview of some common simplifications.*

*Keywords: Model uncertainty, verification, validation, Structural reliability, calibration, Bayesian updating*

## 1. Introduction

The basic uncertainty source considered in Structural Reliability Assessments (SRAs) is the physical uncertainty (also called environmental uncertainty), related to the input non deterministic variables of the physical-mathematical model. Model uncertainty is one among various other uncertainty sources in SRA [1].
Model uncertainty has been considered in deterministic approaches for a long time: many codified design rules include model coefficients (margins). A review of the scientific literature in this area confirms that model uncertainty is not a prevailing research topic in SRA: in SRA the mathematical model of the structural physical

behavior to failure is considered as an input to the analysis, as well as the corresponding deterministic analysis. However, model uncertainty has been a constant concern for the last 3 decades in the SRA researchers' community and is considered with growing interest, especially for risk-sensitive industries; one reason may be that for some structures (dams, big bridges [2]) a failure experience feedback is available and shows that observed failure frequencies may differ significantly from the corresponding calculated failure probabilities.
At EDF, a current example of interest for model uncertainty is the SICODYN research project [3-5], gathering the efforts of 13 academic and industrial research partners. The objective of this project is to evaluate and to improve the predictivity of mechanical models, in the field of dynamic analysis of complex industrial structures (identification of modal quantities like eigenfrequencies). The application example is a pump for which two specimen with identical nominal characteristics, located on two EDF classical thermal power plants, are accessible to measurements. On one side a benchmark study enables to assess the variability of measurements (mode shapes and experimental eigenfrequencies) performed by 5 independent partner teams. On the other side another benchmark exercise evaluates the total variability of calculations of the modal quantities performed by 7 independent partner teams taking their own modeling assumptions (e.g. for boundary conditions). Then, different methodologies are proposed and applied to a priori estimate the confidence associated to a dynamical simulation-based prediction, to quantify the total uncertainty of the numerical simulations which includes model uncertainty [3,4] that can be considered as prevailing in this example, and to update the computational model.
This work is a partial presentation of a methodological contribution to the SICODYN project to address the issue of the modeling of model uncertainty.

## 2. Definition, nature and origin of the model uncertainty

The general procedure of Verification & Validation of models has been initiated by the ASME and has spread to many physical areas, resulting in common notions and practices. Some definitions given in [4] can be mentioned to give a basis to the current framework.

<u>Mathematical model:</u> set of mathermatical relationships representing a physical phenomenon consistently with the underlying scientific theories.

<u>Numerical model:</u> discretized version representing the mathematical model on the computer.

<u>Verification:</u> a process that determines if the numerical model obtained by discretization of the mathematical model of the physical phenomenon and the concerned computer code can be used to represent the mathematical model with sufficient accuracy. Model verification aims to answer the question: ‘‘Are we solving the equations right?’’ [6].

<u>Validation:</u> a process that determines if a model for a physical phenomenon represents the real physical phenomenon with sufficient precision, from the perspective of the end use of the model. This step responds to the following question:

"Are we solving the right equations?" [6]. The validation phase may include a comparison of the simulations with experimental data results. It is mainly relative to the physic-mathematical model.

One can add an extra step: the model calibration, possibly followed by another validation step. This model calibration step consists in introducing parameters (variables) in the model and optimizing them so that the calibrated model be more in agreement with experiment of reference (in general). It is designed to reduce the uncertainty of the physic-mathematical model.

Note, however that this fidelity to the experimental data is sometimes considered inadequate to ensure the credibility of a model and in particular to prejudge its representativeness beyond its strict sphere of validity.

One can see that some of the sources of model uncertainty are voluntary simplifications (voluntarily omitted variables, linearity assumption, neglected interactions), which makes them reducible at least in theory but not in practice necessarily. Thence follows the mixed nature of model uncertainty. These voluntary simplifications may be due either to a search of good compromise between faithful representation of reality and industrial feasibility of the corresponding probability study, either to a desire for conservatism of model combined with a more simple formulation. In the case of willfulness, it rather corresponds to a model error, to distinguish from epistemic uncertainty resulting from an unavoidable lack of knowledge.

This paper focuses on the mathematical model, since the numerical uncertainty is often considered as lower than the physic-mathematical uncertainty. Consequently, it is more concerned with validation than verification.

Two main sources of uncertainty relating to physic-mathematical modeling were identified in the literature and are commonly allowed:

- hidden or voluntarily omitted variables (reduced dimension);
- approximate mathematical formulation (e.g assumption of linearity, omission of the structural dependences between variables (cross effects), discarding local or particular physical phenomena).

This paper investigates the variety of existing representations of model uncertainty in structural reliability models. The general frame is the multiplicative or additive adjustment factor. A simple example is the model coefficient(s) used in the deterministic approach (e.g. codes and standards in civil engineering). In the semi-probabilistic approach, these coefficients can be directly related to corresponding uncertain variables.

# 3. A general modeling framework

Mathematical modeling is present in all areas of science and engineering, in particular in the field of risk analysis, where the model is a simplified representation of a particular aspect of a complex reality. In this area, the mathematical models are used to predict some properties (and their development) of the considered systems [7], and model uncertainty sources are similar to those identified above in Structural Reliability: limitations in knowledge of the phenomenon, deliberate simplifications.

A general form of quantified model uncertainty is proposed [7, 8], this is the approach by adjustment factor, however limited to the case of not deliberate simplifications. The principle of this approach is to use the best available model, noted Y*, and compensate for the error by the introduction of an adjustment factor E.

This adjustment factor can be additive ($E_a$) ou multiplicative ($E_m$). If Y denotes the real output, it comes:

$$Y = Y^* + E_a \quad \text{or} \quad Y = Y^* . E_m \tag{1}$$

In the case where we consider that Y* corresponds in fact to a vector of outputs of the model itself (e.g. the limit state function G (X)), and that this output is probabilistic, it is justified, if one accepts this type of representation, to assume that the adjustment factor is also probabilistic.

# 4. Deterministic approach: example of the model coefficient in Civil Engineering

## 4.1 General notion of safety factor

The safety factor is a concept linked to the deterministic approach: it covers the residual uncertainties on a global basis (or at least some of them), residual uncertainties being those which are not already included in the choice penalizing representative values of the model input quantities considered as variable. As such, it is sometimes called coefficient of ignorance. It is a widespread concept used in many different industrial areas.

In fact, the safety factor should be noted as primarily a number validated by the experience which, associated with a choice of data, a scenario of default and a rule of design, leads to a generally satisfactory design [9]. Indeed, the safety factor is associated with reference values (or even representative values) defined more or less accurately. These values are generally called characteristic values. On these representative values the safety factor is applied. In the basic case Resistance / Load effect (load acting on the structure), the rule of structural verification can be written:

$$R_d > S_d$$

where $R_d$ and $S_d$ are the design values of resistance and load effect (effect of action).

Equivalently, this rule can be written:

$$R_k / S_k > \Theta_k$$

where $R_k$ and $S_k$ are representative (characteristic) of resistance and load effect, and $\Theta_k$ is the safety factor associated with these values.

### 4.2 Particular meaning of the model coefficient

For a long time, deterministic approaches have been taking account of model uncertainty through model coefficients (applied to resistance and/or load variables), which are sometimes included in standards or regulatory procedures.
For some, model uncertainty covers also, besides the adequacy between the predictions of a model and the reality of the behaviour of nature, measurement uncertainties [10]. In this case, the model coefficient should also cover this type of uncertainty. In addition, it should be noted, at least in some cases, that in practice the model coefficient is supposed to cover 'some' uncertainties little or no quantifiable (and often heavily represented in a probabilistic way), considered as mainly due to the human factor [10]. However, taking into account or reducing these uncertainties are normally done by other methods (quality assurance, training, professional documentation, rules of operation...). Finally, it can be noted that when considering a limit state defined as a threshold crossing (type $E \leq C$ where E refers to the effect of the actions), the value of the C threshold generally takes into account the model uncertainty [10].
It comes that the model coefficient is not rigorously defined in the deterministic approach: it can in practice cover uncertainties that are not due to modeling, and model uncertainties may be covered by other methods.

## 5. Model uncertainty: a probabilistic representation is justified

This part refers to early works of Ditlevsen [11], which can be considered as a theoretical reference in structural reliability on the problem of model uncertainty, in any case the oldest received up to us. A precise argument is developed to justify to represent uncertainty in model in probabilistic form, with the aim of a pragmatic model always using the judgment of engineer and professional practices. This argument considers that, for each of the two sources of model uncertainty mentioned at §2, it is possible to represent its impact in a probabilistic manner.

Indeed, the idealized problem corresponds to a formulation based on a limited number of physical random basic variables (dimension n) and a mathematical formulation of the limit state function. For the first source of uncertainty (limitation of the number of random variables), we can consider that the limit state surface separating failure set and safe set corresponds to a particular realization of the hidden variables that have been neglected in the analysis (because they were considered as little important, or that their role is ignored (and therefore presumably limited)). For another realization of these variables, the limit state surface will be another surface slightly different from the idealized surface selected. The 'real' limit state surface can therefore be seen as an unknown disturbance of the idealized surface for which occurs analysis. The evaluation of model uncertainty consists in getting information about all the possible disturbances (i.e. a field of possible variation for the limit state surface).

The position of the 'real' limit state surface is so random, and Ditlevsen [11] proposes a pragmatic approach to evaluate the associated law of probability, since the experimental verification of this uncertainty is almost impracticable beyond the second order moment. An illustration of this reasoning is given on figure 1.

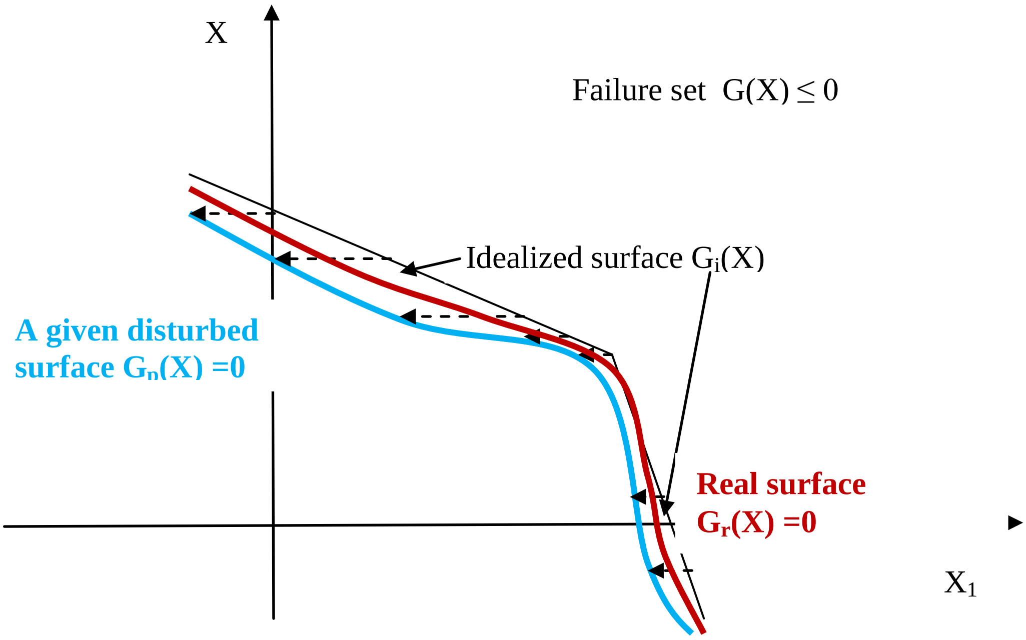


Figure 1 – Representation of the idealized limit state surface, of a given disturbance of this surface, and of the real surface (supposed to be known in this case)

An identical reasoning seems to apply at first sight also for the 2nd source of model uncertainty, i.e. the model uncertainty from the idealized selection of the mathematical expression of the model (e.g. linearity, ignored interactions), in any case when the real form of the model is not known. The lack of knowledge on the 'real' model prompts there also to consider the 'real' limit stat surface as an (unknown) disturbance of the idealized surface. However, it should be noted that, in practice, there are a finite number of potentially relevant models, particularly when assessments by benchmark comparisons are made; in this case, the probabilistic description (at least in the form of a continuous distortion of the surface of failure) is no longer suitable. However, one can also consider that the potential number of acceptable models is high, because these engineering models have sequences of assumptions subject potentially to a combinatorial explosion [12]; in this case it becomes possible to consider the use of a probabilistic representation. However, it should be noted that even in this case, the issue of the selection of model can lead to dismiss many of these 'variants' of a priori possible models.

## 6. Some probabilistic representations

The review presented in this section is not exhaustive. It is limited to two popular expressions. In particular, some simplified versions exist.

### 6.1 First common expression: corrective term including a bias

This expression can be found in [13, 14] for a scalar model (namely in the case of a structural seismic capacity, but the expression is general). It reads:

$$Y(X,\boldsymbol{\Theta}) = G(X) + \gamma(X, \boldsymbol{\theta}) + \sigma_{mod} \,.\, \varepsilon_{mod} \qquad (2)$$

where :

- $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$ denotes the set of the unknown model parameters included in the corrective term;
- $\varepsilon_{mod}$ denotes a random variable (centered and standardized) representing the residual error term;
- $\sigma_{mod}$ denotes the standard deviation of the model error;
- $\boldsymbol{\Theta} = (\boldsymbol{\theta}, \sigma_{mod})$ denotes the comprehensive set of the unknown parameters;
- $\gamma(X, \boldsymbol{\theta})$ denotes the corrective term representing the bias inherent to the deterministic model, considered as a function of the basic model variables X and of the $\theta_i$ parameters;
- $G(X)$ denotes the deterministic selected base model.

This expression suits well to the general tendency where new model development is avoided, and where the idealized model, in general a commonly accepted model, is considered as a basis, to which a corrective term can be added.
However, as will be seen in the sequel, the term $\gamma(X, \boldsymbol{\theta})$ may in fact include terms that appear in the function $G(X)$ (for example to test whether some effects have been correctly estimated).
This expression can be generalized without difficulty in the multidimensional case ($\sigma$ is then a covariance matrix). The vector X of the random variables of the deterministic model database chosen can generally be broken down into two sub-vectors X = (r, s), where r is a vector containing variables of 'resistance' type (i.e. which increases lead to an increase of reliability), and s load type variables (i.e. which increases lead to a decrease in the reliability).

It should be noted first that the unknown $\theta_i$ parameters involved in the correction term are actually the coefficients assigned to functions supposed to explain the systematic error; their choice is necessarily subject to a certain subjectivity [15].

This representation makes the hypothesis of homoskedasticity ($\sigma_{mod}$ independent of X). $\varepsilon_{mod}$ is generally considered to be a normal variable. Furthermore, the authors suggest to use as a correction term $\gamma(X, \boldsymbol{\theta}) = \sum_{i=1}^{p} \theta_i \,.\, h_i(X)$.

This introduces an assumption of linearity in $\theta_i$ parameters. This also implies that it is possible to separate the $\theta$ parameters and the basic X variables; this hypothesis, given the difficulty in gathering sufficient to define with precision the corrective term

($\gamma$ expression), seems quite pragmatic. The number p of parameters must be as limited as possible, in order to have a precision of the estimators as good as possible. Functions $h_i$ are chosen by expert judgement based on the underlying physics-related items and possibly function of experimental data. It can be, as in [13]:

- Elementary terms appearing in the $G$(X) function;
- A constant term (equal to 1) corresponding to a systematic bias ;
- Some basic random variables X ;
- Terms built from random basic variables X and having a precise physical meaning.

In addition, it may be desirable to give these different terms a dimensionless character. It should be noted that the $h_i$ functions can perfectly be correlated (i.e. contain common variables), while the $\theta_i$ parameters or their estimators can also be correlated (statistically).

It is desirable also that the final expression of the model retains a certain simplicity and is consistent with the deterministic model of base; it may however be appropriate to apply an iterative procedure, and to leave at the beginning with a relatively large number of terms. Thus, at the beginning of the procedure, Gardoni [13] goes with a number of p $h_i$ functions greater than the dimension of the deterministic model (i.e. the X vector). The terms of lesser impact are then removed sequentially. A criterion to eliminate terms is to compare the variability of the estimator of parameter $\theta_i$ with $\sigma_{mod}$. If this variability is too high, the corresponding term can be removed, especially if this removal does not significantly increase $\sigma_{mod}$.

In terms of type of uncertainty, it should be noted that $\varepsilon_{mod}$ represents two types of uncertainty:

- the uncertainty related to the omission of variables in the expression of the model, (irreducible: aleatory uncertainty), unless the omission is deliberate and can be reviewed;
- the uncertainty associated with residual error due to the incorrect form of the model after correction, type epistemic (assumed reducible); the model having undergone a correction, we can assume that this contribution is of lesser importance compared to the previous contribution.

However, in practice it is difficult to distinguish between these two types of uncertainties.

## 6.2 Second common expression: unknown parameters integrated to the physical model

Alternatively, the unknown parameters are sometimes integrated to the physical model instead of the corrective term. Such an expression can be found in [5] and [6], where it has been used for a detailed application. It reads:

$$Y(X) = G(X, \boldsymbol{\theta}) + \gamma(X) + \varepsilon(X) \qquad (3)$$

Where:

- $X$ denotes in [6] the control variables (i.e. the variables that can be controlled during an experiment) : these variables are not updated during bayesian updating ; it should be noted that one can extend the definition to the case of the observable or measurable variables because one can have exactly the same type of validation/calibration problems addressed and the same statistical answers if X cannot be controlled but at least observed;
- $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$ denotes the set of unknown parameters (variables) of the model, used for calibration;
- $Y(X)$ denotes the measurements;
- $G(X, \boldsymbol{\theta})$ denotes the deterministic selected base model (prediction), including the unknown parameters;
- $\gamma(X)$ denotes the systematic bias, assumed to depend only on X;
- $\varepsilon(X)$ denotes the experimental error, supposed to be a Gaussian centered variable.

Note that in [6] this representation applies to the measurements Y(X), and consequently it is not completely comparable to the expression of last paragraph 6.1 which considers the reality without measurement error. However, it could apply in the same way to the reality.

Note also that for a more general purpose, in this case where the measurements Y(X) are considered, the error term should in fact include experimental and model error, and be the sum of the two different terms $\varepsilon(X) = \varepsilon_{exp}(X) + \varepsilon_{mod}(X)$. However, these terms may be difficult to identify separately and this may be the reason why they are not distinguished in this expression. It can be noted also that here, this error variable is supposed to be dependent on the control variables X.

However, the most significant difference between expressions (2) and (3) of paragraphs 6.1 and 6.2 is that in expression (3), the idealized physical model can be calibrated and not (only) a corrective term. Consequently, as a synthesis of both expressions, it is suggested in this paper to propose a more general expression, in which the parameters to be calibrated could be both in the physical model (physical parameters) and in the corrective term (statistical parameters). This expression could read:

$$Y(x) = G(x, \theta_G) + \gamma(x, \theta_\gamma) + \varepsilon\,(x, \theta_\varepsilon) \qquad (4)$$

Where:

- $\theta_G$ denotes the set of physical parameters to be calibrated (physical model)
- $\theta_\gamma$ denotes the set of statistical parameters to be estimated (corrective term)
- $\theta_\varepsilon$ denotes the parameters of the conditional law of ε given X

The interest of this expression could be to separate identification problems.

### 6.3 Some common simplifications or assumptions

1 Frequently, the model is assumed to be unbiased, only the residual error is to be estimated and for this term the following situations appear:

- covariance matrix of the error (multidimensional model);
- diagonal terms equal to zero;
- Homoskedasticity: constant terms of the covariance matrix.

2 Almost systematically, the error term (which can include model error, experimental error or both) is assumed to follow a normal distribution (or lognormal in the case of a multiplicative term). No exception was encountered.

3 In consistence with codified rules, the basic deterministic model is sometimes very simplified. Kaminski [16] presents a case where the model is the basic case R – S, with R and S assumed to follow a Gaussian distribution. The only issue is then to estimate the variation coefficient of the model error.

## 7. Conclusion

For a long time, deterministic approaches have been taking account of model uncertainty through model coefficients (applied to resistance and/or load variables), which are sometimes included in standards or regulatory procedures. However, the model coefficient is not rigorously defined in the deterministic approach: it can in practice cover uncertainties that are not due to modeling, and model uncertainties may be covered by other methods.
It has been reminded in this paper that it is acceptable (although not a unique possible framework) to propose a probabilistic representation of model uncertainty, thanks to the original justification proposed by Ditlevsen: the “real” limit state surface can be considered as a random, unknown disturbance of the idealized surface used for the analysis.

A variety of probabilistic modelings for model uncertainty appear in the literature, and some of them have been presented in this paper. Their equivalence is not automatic; its investigation is out of the scope of this paper. However, some common issues can be pointed out. All modelings include unknown parameters as well as a residual error term. These parameters, likely to be calibrated to improve the predictability of the model, are incorporated either to the base model, or, in the most common representation, to a corrective term representing the systematic bias of the base model. It could be suggested to propose a more general representation, where the parameters to be calibrated could be both in the physical model (physical parameters) and in the corrective term (statistical parameters). Depending on the representations, the residual error is either a model error, or a mixed term combining model error and experimental, or just an experimental error. Various simplifications are encountered: Gaussian error (no exceptions), unbiased model (no corrective term).

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