A periodic inspection strategy for systems subject to competing failure modes sharing a common risk source

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Abstract

In this paper, a stochastic failure model for a system with stochastically dependent competing failures is analyzed. The system is subject to two causes of failure: degradation failures and catastrophic failures. Both types of failures share an initial common source: an external shock process. Each external shock in its arrival, instantly initiates an independent degradation process in the system. Also, each external shock increases the failure rate function of the distribution of the catastrophic failures. Under this setting, a maintenance strategy is developed. Following this maintenance strategy, the system is inspected periodically and it is correctively replaced at failure and preventively replaced when the degradation level of a degradation process exceeds a critical threshold. The expected cost rate for this maintenance model is analyzed and numerical examples are given that illustrate the analytical results.

Keywords: Degradation process, condition-based maintenance, gamma process, non-homogeneous Poisson process, dependent competing risks

1. Introduction

Most of the systems suffer a physical degradation before they fail. According to Bogdanoff et al. (1985), this degradation is due to the irreversible accumulation of damage through life and may involve corrosion, material fatigue, wearing out and fracturing. Since degradation is a complex random mechanism, it is well represented by a stochastic process frequently stationary and with independent increments (see, e.g., Grall et al. (2002), Liao et al. (2006) and Mercier and Castro (2013)). Under this stochastic-process approach, the system is regarded as failed when its degradation first reaches a critical threshold level. In the literature on degradation-based stochastic failure models, frequently, the degradation process is considered the only cause of failure. However, in many practical situations, systems are not only subject to internal degradation, but also are exposed to catastrophic failures that can provoke a sudden failure (see eg., Huynh et al. (2011), Huynh et al. (2012) and Castro et al (2014)). Furthermore, the causes of failure can be stochastically dependent due to many different reasons.

In this paper, we analyze a stochastic failure model for a system with "stochastically dependent" competing risks and develop a preventive maintenance policy for it. The system is subject to two types of failures: degradation failure and catastrophic failure. The initial common source for these two types of failure is a random external shock process. More specifically, each external shock causes the start of a degradation process of the system and also provokes an increment of the failure rate of the catastrophic failure. Since the same external shock process affects both competing risks, they are stochastically dependent.

The setting of this work is a particular case of the general framework showed by Cha and Castro (2014) where they considered that each external shock provoke, no just the increasing of the failure rate of the catastrophic failures, but the total breakdown of the system. The work of Cha and Castro (2014) was mainly focused on the dependence properties between the failure mechanisms. In the paper presented to this conference, we are more concerned about the maintenance of this system, developing a maintenance strategy with periodical inspections. In an inspection time, if the system is failed, it is correctively replaced. Otherwise, the degradation levels of each degradation process are measured. If one of the degradation processes exceeds a preventive threshold, the system is preventively replaced. The goal is to find the optimal value of the time between inspections and the optimal value of the preventive threshold that minimize an objective cost function.

The structure of this paper is as follows. In Section 2, the assumptions of the model and the stochastic failure model that describes its functioning is described. In Section 3, the preventive maintenance strategy is showed and the long-run average cost rate is obtained. Section 4 shows a numerical example of this maintenance strategy and Section 5 concludes.

2. Assumptions of the model

As we explained in the introduction, in this paper we deal with a system subject to two competing risks (degradation and catastrophic failures) and under an external shock process. This external shock process provokes the starting of different degradation processes and also the increment of the failure rate function of the catastrophic failures. The assumptions of this model are the following.

1. We consider an unitary system working under a dynamic environment. We assume the system is subject to external shocks that arrive to the system according to a non-homogeneous Poisson process $\{N(t),\ t \geq 0\}$ with intensity function $\lambda(t)$ where $N(t)$ denotes the number of external shocks at time t. Let T_i be the arrival time of the *i*-th external shock, $i=1, 2, \ldots$
2. The system is subject to two competing causes of failure: catastrophic failure (Cause I) and degradation failure (Cause II).

3. Let Y be the time to a catastrophic failure with baseline failure rate function given by $r_0(t)$. We assume that each external shock increases the failure rate function of Y in the following way

$$r(t|N(t) = n(u), 0 \le u \le t) = r_0(t) + \eta n(t)$$

where $\eta > 0$ and $t \ge 0$. That is, $r_0(t)$ define the failure rate function of the catastrophic failures in absence of external shocks. Just after an external shock, the failure rate function of Y is increased by η. Hence, the survival function of Y given the external shock process is given by

$$P(Y > t|N(u), 0 \le u \le t) = \exp\left\{-\int_0^t r_0(s)ds - \eta \sum_{j=1}^{N(t)} (t - T_j)\right\}, \quad t \ge 0.$$

4. We assume that each shock instantly initiates an independent degradation process in the system. Let $X_v(t)$ be the deterioration level of the degradation process *t* time units after its initiation point *v*. We assume that $\{X_v(t),\ t \ge 0\}$ follows a gamma process with parameters α and β, that is, the random variable $X_v(t)$ has the following probability density function (p.d.f)

$$f_{X_v}(t) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\beta x}, \ \alpha > 0, \ \beta > 0$$

We assume that a degradation failure occurs when a degradation process exceeds the failure threshold L. Let $\sigma_L(v)$ be the time from v until the degradation, initiated at v, reaches the threshold failure L with survival function given by

$$\overline{G}_v(t; L) = P(\sigma_L(v) > t) = \int_0^L f_{X_v(t)}(x)dx.$$

Let S_c be the time to a degradation failure. The conditional survival function of S_c given the external shock process, is obtained as follows

$$P(S_c > t|N(u), 0 \le u \le t) = \prod_{k=1}^{N(t)} \overline{G}_{T_k}(t - T_k; L), \quad t \ge 0.$$

5. The system is inspected each T time units to evaluate its state. If the system is down in an inspection time, a corrective maintenance is performed and the system is replaced by a new one. If the system is not down at the inspection time, the deterioration level of each degradation process is measured. If the deterioration level of a degradation process exceeds a fixed preventive threshold M (M < L), a preventive maintenance is performed and the system is replaced by a new one. Otherwise, no maintenance task is performed. We assume the time necessary to perform a maintenance action is negligible.

6. The costs of a corrective and preventive maintenance are C_c and C_p, respectively. The cost of each inspection is C_i and the cost incurred by the inactivity of the system is C_d. We assume $C_c > C_p > C_d > C_i > 0$.

Let $\overline{F}_{\min(Y,S_c)}$ be the survival function of the random variable min(Y, S_c). Cha and Castro (2014) obtained the survival function of min(Y, S_c) given by

$$\overline{F}_{\min(Y,S_c)}(t) = \exp\left\{-\int_0^t r_0(s)ds\right\}\exp\left\{-\int_0^t \left(1-\exp\{-\eta(t-x)\}\overline{G}_x(t-x;L)\right)\lambda(x)dx\right\}, \quad t \geq 0.$$

Let S_p be the random variable that describes the time instant at which any one of the degradation processes exceeds the preventive threshold M for the first time. Cha and Castro (2014) obtained the joint distribution of the random variable (min(Y, S_c), S_p). For $0 \leq u \leq t$, we denote by $\overline{F}_1(t,u) = P(\min(Y,S_c) \geq t, S_p \geq u)$ and for $0 \leq t \leq u$ $\overline{F}_2(t,u) = P(\min(Y,S_c) \geq t, S_p \geq u)$.

For $0 \leq u \leq t$, we get

$$\begin{aligned}\overline{F}_1(t,u) &= \exp\left\{-\int_0^t r_0(u)du\right\}\exp\left\{-\int_0^u \left(1-\exp\{-\eta(t-x)\}\overline{G}_x(u-x,t-x;M,L)\right)\lambda(x)dx\right\} \\ &\cdot \exp\left\{-\int_u^t \left(1-\exp\{-\eta(t-x)\}\overline{G}_x(t-x;L)\right)\lambda(x)dx\right\}\end{aligned}$$

where

$$\begin{aligned}\overline{G}_x(u,t;M,L) &= P(X_x(u) \leq M,\ X_x(t) \leq L) \\ &= \int_0^M f_{X_x(u)}(x)F_{X_x(t)-X_x(u)}(L-x)dx,\end{aligned}$$

where $F_{X_x(t)-X_x(u)}(x)$ denotes the cumulative distribution function of the random variable $X_x(t)-X_x(u)$ and, for $0 \leq t \leq u$,

$$\begin{aligned}\overline{F_2}(t,u) &= \exp\left\{-\int_0^t r_0(u)du\right\}\exp\left\{-\int_0^t \left(1-\exp\{-\eta(t-x)\}\overline{G}_x(u-x;M)\right)\lambda(x)dx\right\} \\ &\cdot \exp\left\{-\int_u^t \left(1-\overline{G}_x(u-x;M)\right)\lambda(x)dx\right\}.\end{aligned}$$

In the sequel of this work, we denote by $f_1(t,u)$ and $f_2(t,u)$ the bidimensional density functions associated to $\overline{F}_1$ and $\overline{F}_2$ respectively. That is,

$$\overline{F}_1(t,u) \quad = \quad \int_t^\infty \int_u^\infty f_1(w,s)ds \;\; dw \qquad (1)$$

$$\overline{F}_2(t,u) \quad = \quad \int_t^\infty \int_u^\infty f_2(w,s)ds \;\; dw \qquad (2)$$

3. Maintenance strategy

We assume that the failure of the system can be detected only by an inspection and the system is inspected each T units of time to check its status. On each inspection, if the system is failed, then the system is replaced by a new one (a corrective maintenance). If it is not, in these inspection times, the deterioration level of the degradation processes is measured. If the deterioration level of any degradation process exceeds a threshold M (0 < M < L), then the system is replaced by a new one (a preventive maintenance); otherwise no maintenance task is performed. The time to perform a maintenance action is negligible.

Now, we obtain the probability of the different maintenance tasks.

Probability of a preventive maintenance

Let $P_p(kT)$ be the probability of a preventive maintenance action at time kT. A preventive maintenance action is performed at time kT, k = 1, 2, …, if

$$\left\{(k-1)T \le S_p \le kT, \quad kT \le \min(Y, S_c)\right\},$$

occurs, hence

$$\begin{aligned} P_p(kT) \quad &= \quad P[(k-1)T \le S_p \le kT, kT \le \min(S_c, Y)] \\ &= \int_{(k-1)T}^{kT} du \int_{kT}^{\infty} f_1(t,u)dt, \end{aligned} \qquad (3)$$

for k=1, 2, … and T > 0.

Probability of a corrective maintenance

Let $P_c(kT)$ be the probability of a corrective maintenance action at time kT. A corrective maintenance action is performed at time kT, k = 1, 2, …, if

$$\left\{\min(Y, S_c) \le T\right\}$$

occurs and

$$\left\{(k-1)T < S_p, \qquad (k-1)T < \min(Y, S_c) \le kT\right\},$$

occurs for k=2, 3, … and T > 0. Hence, the probability of a corrective maintenance at time kT is given by

$$
\begin{aligned}
P_c(kT) \;&=\; F_{\min(Y,S_c)}(T)1_{\{k=1\}} + \left(\int_{(k-1)T}^{kT} du \int_{(k-1)T}^{u} f_2(t,u)dt \right. \\
&+ \left. \int_{(k-1)T}^{kT} du \int_{u}^{kT} f_1(t,u)dt + \int_{kT}^{\infty} \int_{(k-1)T}^{kT} f_2(t,u)dtdu \right) 1_{\{k>1\}}
\end{aligned}
\tag{4}
$$

Expected time to a replacement cycle

A replacement cycle is the time between successive replacements of the system. Let R be the time to a replacement cycle under this maintenance strategy. Then

$$
R = \begin{cases} kT & (k-1)T < S_p \le kT, \quad kT \le \min(S_c, Y) \\ kT & (k-1)T < S_p, \quad (k-1)T < \min(Y, S_c) < kT \end{cases}
$$

for k=1, 2, …. Then,

$$
E[R] = \sum_{k=1}^{\infty} kT\left(P_p(kT) + P_c(kT)\right) \tag{5}
$$

where $P_p(kT)$ and $P_c(kT)$ are given by (3) and (4) respectively.

Expected number of inspections

Let N be the number of inspections in a replacement cycle. Hence, the expected number of inspections in a replacement cycle is given by

$$
E\left[N\right] = \sum_{k=1}^{\infty} k\left(P_c(kT) + P_d(kT)\right) \tag{6}
$$

where $P_p(kT)$ and $P_c(kT)$ are given by (3) and (4) respectively.

Expected downtime in a replacement cycle

Let W be the system downtime in a replacement cycle. Then,

$$
\begin{aligned}
E[W] \;&=\; E\left[(T - \min(Y,S_c))1_{\{\min(Y,S_c)\le T\}}\right] \\
&+ \sum_{k=2}^{\infty} E\left[(kT - \min(Y,S_c))1_{\{(k-1)T<S_p,(k-1)T<S_p,(k-1)T<\min(Y,S_c)\le kT\}}\right] \\
&= \int_0^T \left(T - f_{\min(Y,S_c)}(t)\right)dt + \sum_{k=2}^{\infty}\left(\int_{(k-1)T}^{kT} du \int_{(k-1)T}^{u} f_2(t,u)\left(kT - t\right)dt \right) \\
&+ \sum_{k=2}^{\infty}\left(\int_{(k-1)T}^{kT} du \int_{u}^{kT} f_1(t,u)(kT-t)dt + \int_{kT}^{\infty} du \int_{(k-1)T}^{kT} f_2(t,u)(kT-t)dt \right)
\end{aligned}
\tag{7}
$$

where f_1 and f_2 denotes the bidimensional density functions of the joint distribution of the random variable ($\min(Y,S_c)$, S_p) given by (1) and (2) respectively.

Expected cost rate

Let C(T,M) be the expected cost rate for this maintenance model. By Tijms (2003),

$$C(T,M) = \frac{E[C]}{E[R]}$$

where E[C] denotes the expected rate in a replacement cycle and E[R] the expected time between replacements of the system. So,

$$C(T,M) = \frac{C_c \sum_{k=1}^{\infty} P_c(kT) + C_p \sum_{k=1}^{\infty} P_p(kT) + C_I E[N] + C_d E[W]}{E\left[R\right]} \tag{8}$$

where $P_c(kT)$, $P_d(kT)$, E[R], E[N] and E[W] are given by (3), (4), (5), (6) and (7) respectively.

The search of the optimal maintenance strategy is reduced to find the values M_{opt} and T_{opt} such that

$$C(T_{opt}, M_{opt}) = \inf\left\{C(T,M), \quad T > 0, \quad 0 < M \leq L\right\}$$

where C(T,M) is given by (8). Due to the analytical complexity of C(T,M), the optimization of the expected cost rate for a data set is performed in the next section using numerical methods.

4. Numerical examples

In order to illustrate the analytical results, a numerical example is here showed. We consider a system subject to a random external shock process with intensity λ=1/100 external shocks per unit time. These shocks increment the catastrophic failure rate by η = 1/1000. We also assume that the baseline rate function of the catastrophic failure is given by $r_0(t)$ = 1/200 catastrophic failures per unit time. Each shock, in its arrival, initiates a degradation process. The deterioration level of this degradation process is modelled by a gamma process with parameters α = 1 and β = 3. Then, the deterioration level t time units after its initiation point v, $X_v(t)$, follows a gamma distribution with parameters t and 3, having the corresponding pdf

$$f_{X_v}(t) = \frac{e^t}{\Gamma(t)} x^{t-1} e^{-3x}$$

for t ≥ 0. The cost of a corrective replacement is 10 monetary units (m.u), the cost of a preventive replacement is 8 m.u., the cost of an inspection is 0.005 m.u. and the downtime cost is 0.5 m.u per unit time. Figure 1 shows the expected cost rate versus

T and M. The values of this figure have been calculated using Monte Carlo simulation with 150 values of T and 40 values for M and 60000 simulations for each pair of values. The optimal values for T and M are given by T_{opt} = 5.9 and M_{opt} = 6.9231 with an optimal expected cost rate of $C(T_{opt},M_{opt})$= 0.0644 monetary units per unit time.

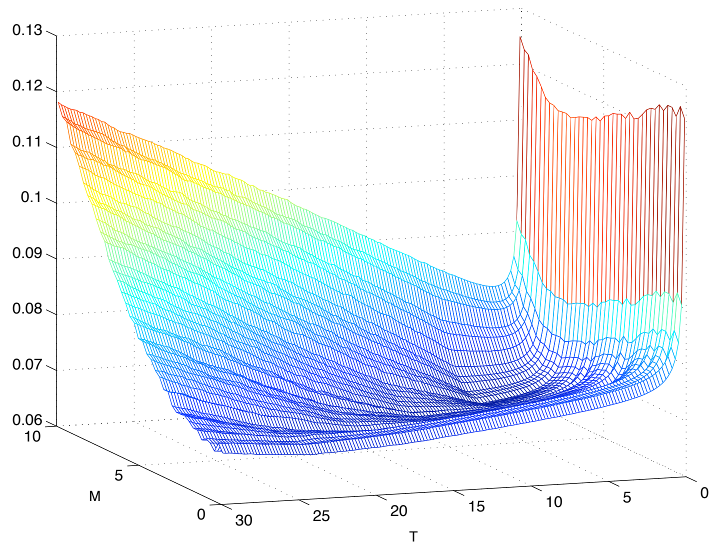

Figure 1. Expected cost rate versus M and T

5. Concluding remarks and further works

In this paper, we have studied a maintenance strategy for a system under a random shock environment with stochastically dependent competing risks. While the system maintenance with 'independent competing risks' or with very simple dependency structure has been studied under specific settings in the literature (see, e.g., Huynh et al. (2011), Huynh et al. (2012), Castro et al. (2013)), this work is pioneering in considering the dependent competing risks models sharing external shocks under a general setting.

In this paper, the preventive maintenance is based on the deterioration level of the degradation process. An extension of this paper would be to consider another type of maintenance strategy, for example, a preventive maintenance based on the mean residual life of the system.

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