Stochastic finite elements: a B-spline approach

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Abstract

The paper presents a framework for the uncertainty analysis of structures characterized by random properties or subject to random loads. These uncertainties are modeled by a set of random variables. The novelty of the paper consists in the use of B-spline basis functions for the approximation of the uncertain structural response. B-spline functions are widely used in computer aided engineering and computer aided manufacturing, because they are flexible and accurate tools for the representation of surfaces. The potentiality of the approach is discussed in an application example, which concerns the radial plastic displacement of a hollow sphere under internal pressure and the resistance of a reinforced cross section in flexure. The accuracy of the approach is compared with the Polynomial Chaos Expansion-based stochastic finite element method and the Stochastic Collocation method.

Keywords: Stochastic FEM, Polynomial Chaos Expansion, Stochastic Collocation method, B-spline basis functions.

1. Introduction

The uncertainty in the response is related to the material properties, geometrical dimensions, environmental conditions and mathematical models that are used to predict the structural performance. Under these circumstances, a probability-based approach is suggested for the structural assessment and uncertainty analysis. The uncertain parameters are usually described by random variables. However, random fields can be efficiently used to model the random spatial variability of the uncertain parameters [1-3].

The random response of the structure is often evaluated by means of the Monte Carlo method or explicitly represented by the Stochastic Collocation method [4] or by the Polynomial Chaos Expansion (PCE) [5]. According to the Monte Carlo method, the statistics of the structural response can be estimated by performing several structural analyses with samples drawn from the distribution of the input random parameters. This approach has been used in several applications, due to its simplicity and robustness. However, a large number of simulations is necessary in order to obtain accurate solutions, due to the slow rate of convergence of the Monte Carlo method.

According to the Stochastic Collocation method, the response of the structure is approximated by means of Lagrange interpolation of the structural response at a suitable set of realizations of the input random parameters. According to the PCE approach, the response is written as a series expansion of Hermite polynomials of standard normal random variables. The coefficients of the PCE can be estimated by means of intrusive [5] and non-intrusive methods [6-7]. Whatever method is used, the number of structural analyses dramatically increases with the number of input random variables and truncation order of the PCE.

A numerical approach based on the SFEM and B-spline functions is presented in the paper for the uncertainty analysis of the response of mechanical systems. The approach relies on the approximation of the random structural response by means of B-spline surfaces.

2. Polynomial Chaos Expansion

The response Y can be expressed in terms of Hermite polynomials in standard normal random random variables [5]:

$$Y = a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1\left(\xi_{i_1}\right) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2\left(\xi_{i_1},\xi_{i_2}\right) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3\left(\xi_{i_1},\xi_{i_2},\xi_{i_3}\right) + \ldots \tag{1}$$

where $\{\xi_i\}_{i=1}^{\infty}$ denotes a set of standard normal random variables, $\Gamma_p\left(\xi_{i_1},\xi_{i_2},\ldots,\xi_{i_p}\right)$ is the polynomial chaos (PC) of order p and $\{a_i\}_{i=1}^{\infty}$ are deterministic coefficients. The PC of order *p* is the set of all orthogonal Hermite polynomials of degree not exceeding *p* in the random variables $\{\xi_i\}_{i=1}^{\infty}$. The PCE has been written in Eq. (1) in terms of Hermite polynomials and standard normal random variables. However, other polynomials can be used for non-Gaussian random variables, through the Askey-scheme of polynomials [8-9].

The input random variables $\{X_i\}_{i=1}^{\infty}$ (which represent material properties, geometrical dimensions, etc.) can be related to the random variables $\{\xi_i\}_{i=1}^{\infty}$ [10] by means of an iso-probabilistic transformation. If the random variables are independent, this transformation is:

$$\xi_{\mathrm{i}} = \Phi^{-1}\left[\mathrm{F}_{\mathrm{X_i}}\left(x_i\right)\right] \tag{2}$$

where F_{xi} is the cumulative distribution function of the random variable X_i. For simplicity of the notation, Eq. (1) is rewritten as:

$$\mathrm{Y} = \sum_{\mathrm{i}=1}^{\infty} \hat{\mathrm{a}}_{\mathrm{i}}\psi_i(\xi) \tag{3}$$

where there is a one-to-one correspondence between the coefficients and between Γ and Ψ. In the practical applications, the number of random variables is finite and PCs of low order are considered. Usually linear, quadratic, cubic polynomials are used, since they provide a compromise between accuracy of the approximation and its computational cost. As a result, the PCE expansion is truncated at the P-th term:

$$\widetilde{\mathrm{Y}} = \sum_{\mathrm{i}=1}^{\mathrm{P}} \hat{\mathrm{a}}_{\mathrm{i}} \psi_i(\xi) \tag{4}$$

where P depends on the number n of random variables and on the PC order p:

$$\mathrm{P} = \frac{(\mathrm{n}+\mathrm{p})!}{\mathrm{n}!\mathrm{p}!} \tag{5}$$

The PCE is the basis of the spectral stochastic finite element method (SSFEM), which was developed by Ghanem and Spanos [5] for mechanical problems involving random fields. According to the SSFEM, the coefficients of the PCE in Eq. (4) are computed using a Galerkin scheme. This approach is usually called “intrusive”, since the system of equation of the deterministic FEM is modified in order to account for the randomness in the input random variables. Moreover, the resulting system of equations is much larger than the one from deterministic finite element analysis.

As an alternative, non-intrusive computational methods have been recently developed. The coefficients of the PCE expansion are computed by evaluating several times the response Y. Two methods can be distinguished with respect to the approach used in the estimation of the coefficients: the spectral projection and the regression methods [7]. In the spectral projection method, the coefficients of the PCE are formulated as multi-dimensional integrals, which result from the projection of the response Y on the PC. These integrals can be computed by means of quadrature rules or by simulation. By means of the regression method, the PCE coefficients are computed by minimizing the mean square error of the random response. Given m sample points, the regression method consists in finding the coefficients of Eq. (4) that minimize the difference between the true response and the PC approximation:

$$\varepsilon = \sum_{i=1}^{m} (\mathrm{y}_{\mathrm{i}} - \widetilde{\mathrm{y}}_{\mathrm{i}})^2 \tag{6}$$

where y_i and $\widetilde{y}_i$ are the realizations of the true response Y and PCE expansion of Eq. (4). This optimization problem leads to a linear least square problem. Suggestions for the choice of the sample points are available in literature. The Latin Hyper Cube Sampling (LHS) is suggested in [9], while the roots of the Hermite polynomial of one order higher than the PC order are selected as sampling points in [11, 12]. In this case, the use of a tensor-product sampling scheme leads to a number of sampling points which is usually greater than the number of the unknown coefficients of the PCE. It is important to notice that the number of sampling points increases with the number of input variables and the order of expansion. Hence, the LHS could be computationally more efficient even for low dimensional problems.

Calculating the PC coefficients is the most important step of the approximation of the response by means of Eq. (4). The computational aspect is particularly important in the context of the spectral projection and regression methods.

3. Stochastic Collocation method

The stochastic collocation method relies upon the Lagrange interpolation of the system response. Therefore the problem is to find an accurate approximation of the response $Y(\xi)$ by means of Lagrange interpolation of the values of the response at selected outcomes of the random vector ξ. As an example, if the problem involves only one variable, the approach leads to the following approximation:

$$\tilde{Y}(\xi) = \sum_{i=1}^{M} y^{(i)} L_i(\xi) \tag{7}$$

where $\tilde{Y}(\xi)$ is the polynomial approximation of the system response, M is the number of interpolation points, $y^{(i)}$ is the value of the system response at the i^{th} interpolation point and $L_i(\xi)$ is a Lagrange interpolation polynomial, which is defined as follows:

$$L_i(\xi) = \prod_{\substack{j=1 \\ j \neq i}}^{M} \frac{\xi - \xi^{(j)}}{\xi^{(i)} - \xi^{(j)}} \tag{8}$$

An advantage of this method is the ease of implementation. Once the grid of interpolation points has been chosen, the system response is evaluated by a deterministic model. The system response is then approximated by means of Eq. (7) and the statistics of the response can be obtained by post-processing operations. In case of multi-dimensional random vectors ξ, tensor product interpolation is applied. If tensor product grids are used, the number of response evaluations grows exponentially with the dimension of vectors ξ. Therefore, this method suffers from the curse of dimensionality, unless Smolyak formulae [13] are used.

4. B-spline SFEM

4.1 B-spline functions

B-splines are polynomial functions which can be used to describe curves and surfaces. NURBS, which are derived from the B-splines, are the standard tools in computer aided design and computer graphics [14-15]. A B-spline curve is written as a linear combination of *n* B-spline basis functions and the corresponding control points:

$$C(t) = \sum_{i=1}^{n} B_i N_{i,p}(t) \tag{9}$$

where p is the degree of the B-spline basis functions (p=0, 1, 2, 3 refer to constant, linear, quadratic and cubic functions) and u is a parameter which is defined over an interval $[u_{min}, u_{max}]$. B-splines basis functions are defined recursively starting from piecewise constant functions:

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \le t < t_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

The values u_i belong to the knot vector $\mathbf{T}=[t_1, t_2, \ldots, t_{n+p+1}]$, which is a set of coordinates. The so-called open uniform knot vector is used in this paper. The first and last values of the knot appear p+1 times, while the other values of t are equally spaced. As an example the non-zero B-spline basis functions of degree 0 corresponding to the knot vector [0,0,0,1,2,2,2] are plotted in Figure 1.

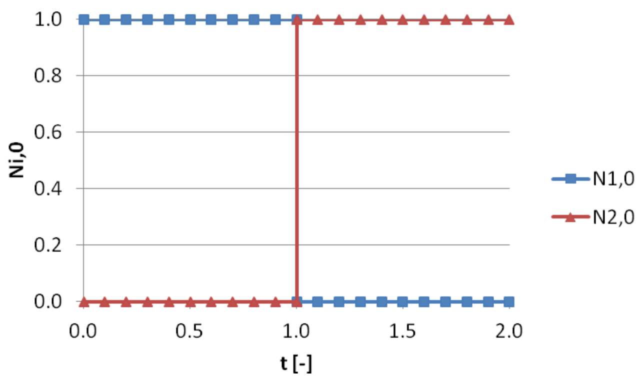

Figure 1. B-spline p=0.

The B-spline basis functions of higher order are obtained as follows:

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t) \tag{11}$$

As an example, the B-spline functions of degree 1 and 2 plotted in Figures 2 and 3 are obtained from the functions of Figure 1 by means of Eq. (11).

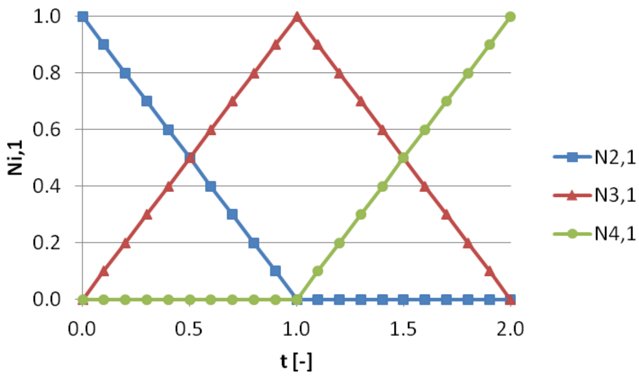

Figure 2. B-spline p=1.

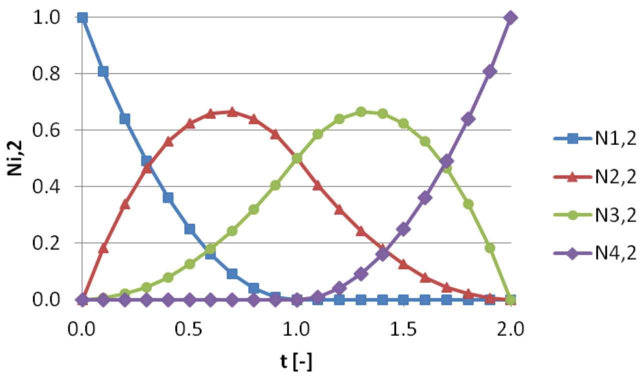

Figure 3. B-spline p=2.

B-spline curves (Eq. (9)) have two important properties. First, they have a non-global behavior because each control point is associated with a unique basis function. The support of each basis function $N_{i,p}(t)$ is the interval $[t_i, t_{i+p+1}]$. It means that only a limited portion of a B-spline curve is affected if the position of a control point changes. Second, the degree *p* of the B-spline basis can be changed without changing the number of control points *n*, provided that *p* is less than *n*.

B-spline basis functions can be used also to represent surfaces:

$$S(t,v) = \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ij} N_{i,p}(t) N_{j,q}(v) \tag{12}$$

where n and m are the numbers of basis functions in the t and v directions and the degree p and q can be chosen independently. This approach can be extended to functions in several variables.

4.2 Representation of the uncertain response

For simplicity, the procedure is explained for the one-dimensional case. The response $Y(\xi)$ is approximated by means of Eq.(9):

$$\tilde{Y}(\xi) = \sum_{i=1}^{n} B_i N_{i,p}(\xi) \tag{13}$$

This approximation requires two steps. First, the response at N > n sample points is evaluated. Then, the coordinates of the control points are determined by minimizing the difference between the response at the sample points and its B-spline approximation. Hence, the coordinates of the control points are the solution of the following least square problem:

$$\mathbf{B} = \mathrm{argmin} \sum_{i=1}^{N} \left(y^{(i)} - \tilde{y}^{(i)} \right)^2 \tag{14}$$

where $\tilde{y}$ is the B-spline approximation.

The number N of data points and their position influences the efficiency and accuracy of the approximation. Only uniformly distributed points are considered in this work. It is expected that the method could suffer from the curse of dimensionality because the number of evaluation of the response Y is equal to N^n where n is the number of random variables. The N data points in each variable should be distributed on a wide domain, in order to properly account for the dependence of the response Y to each random variable.

5. Application example

In the following, the Polynomial Chaos Expansion method, the Stochastic Collocation method and the proposed B-spline SFEM are compared in terms of the mean value, standard deviation, skewness and kurtosis of the radial plastic displacement of a hollow sphere under internal pressure [16] is investigated. Third order PCs and cubic B-splines are used in this example, as they usually provide accurate results.

In the paper of Baroth et al. [16], the accuracy of the Stochastic Collocation method is investigated by means of a parametric analysis. A tensor product interpolation grid with N=9, 25, 36 and 100 points is considered. The LHS method is used to generate N samples for the estimation of the PCE coefficients and the control polygon vertices in the B-spline-based SFEM.

A hollow sphere under internal pressure (see Fig. 4) is considered. The material is assumed to be elastic perfectly plastic. The plastic radial displacement at the inner outline is governed by the inner and outer radii a and b, the depth of the plastic zone c, the elastic modulus E_s, the Poisson ratio ν, the yield stress f_y and the pressure p.

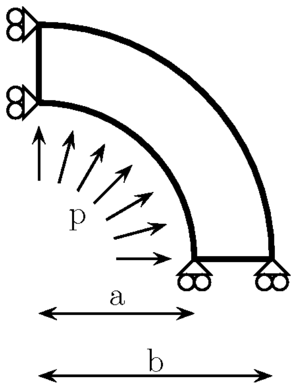

Figure 4. Sphere under internal pressure.

A closed form expression for the radial plastic displacement at the inner outline is available:

$$u = \frac{af_y\left[(1-\nu)\frac{c^3}{a^3} + 2(2\nu-1)\left(\ln\left(\frac{c}{a}\right) + \frac{1}{3}\left(1-\frac{c^3}{b^3}\right)\right)\right]}{E_s} \tag{15}$$

where c is the solution of the following equation:

$$p = 2f_y\left[\ln\left(\frac{c}{a}\right) + \frac{1}{3}\left(1-\frac{c^3}{b^3}\right)\right] \tag{16}$$

The elastic modulus E_s and the Poisson ratio ν are described by means of two lognormal random variables:
- $E_s \sim LN(2\cdot 10^{11}, 6\cdot 10^{10})$ Pa;
- $\nu \sim LN(0.3, 0.03)$.

The correlation coefficient between the two random variables is equal to 0.8. The other parameters of Eq. (15) are assumed as deterministic:
- $a=1\cdot 10^{-3}$ m;
- $b=2\cdot 10^{-3}$ m;
- $f_y=3\cdot 10^{8}$ Pa;
- $p=3.589\cdot 10^{8}$ Pa.

The radial displacement (Eq. (15)) is plotted in Fig. 5 as a function of the standard normal random variables ξ_1 and ξ_2, which are obtained by an iso-probabilistic transformation of the random variables E_s and ν.

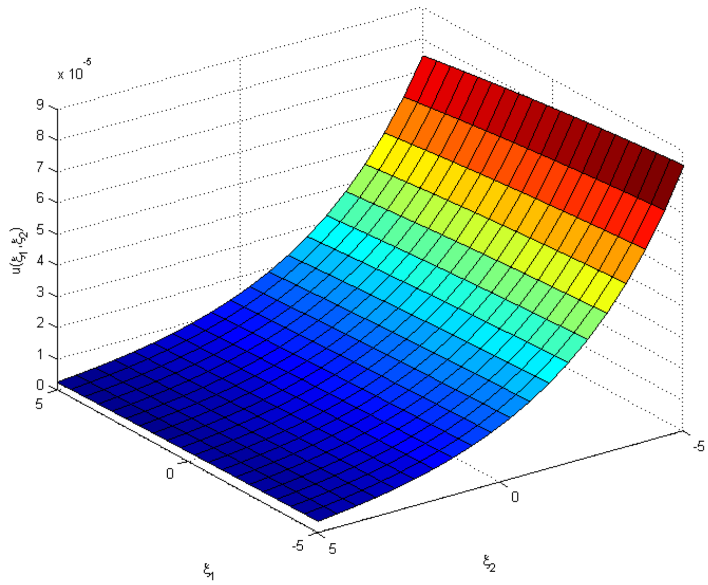

Figure 5. Response.

The relative errors on the first four statistical moments, with respect to the values estimated with 50000 Monte Carlo simulations, are plotted in Figs. 6-9.

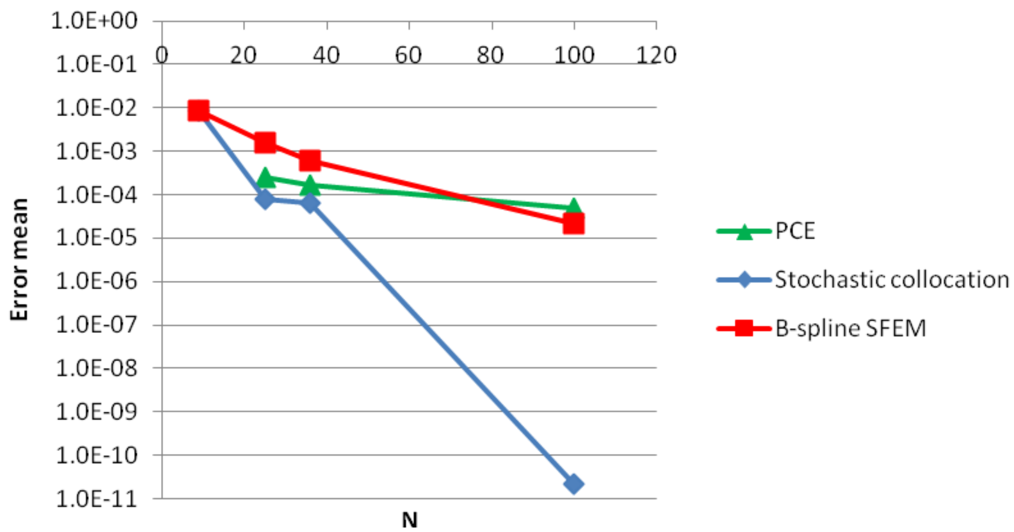

Figure 6. Error on the mean value.

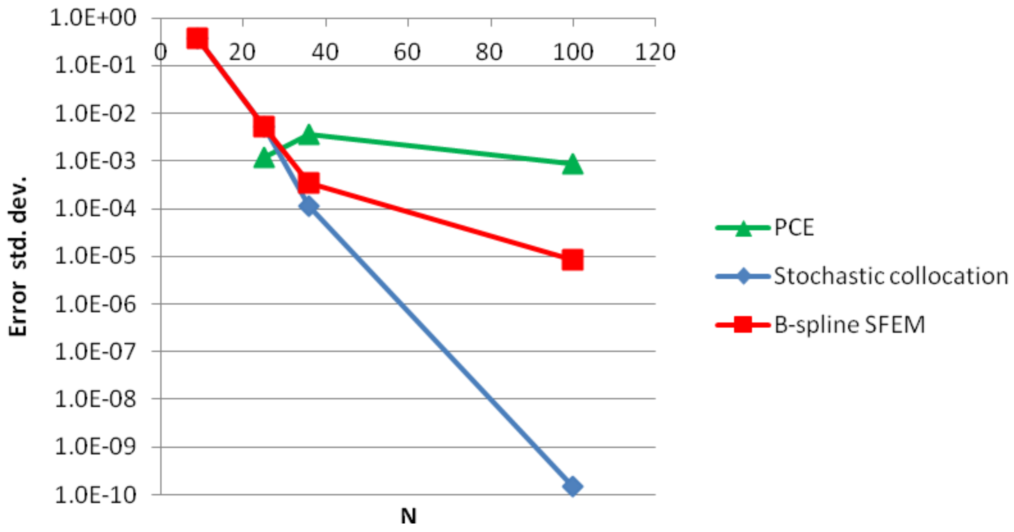

Figure 7. Error on the standard deviation.

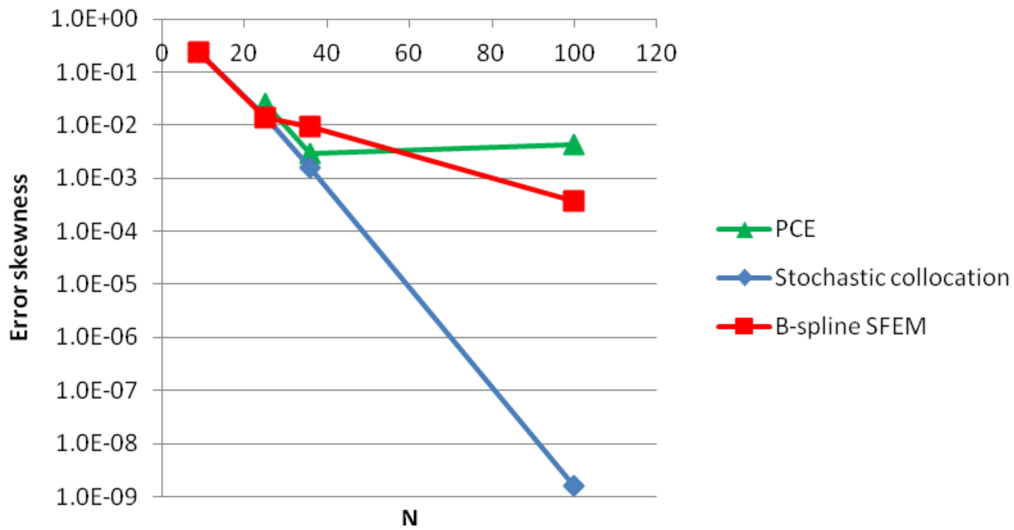

Figure 8. Error on the skewness.

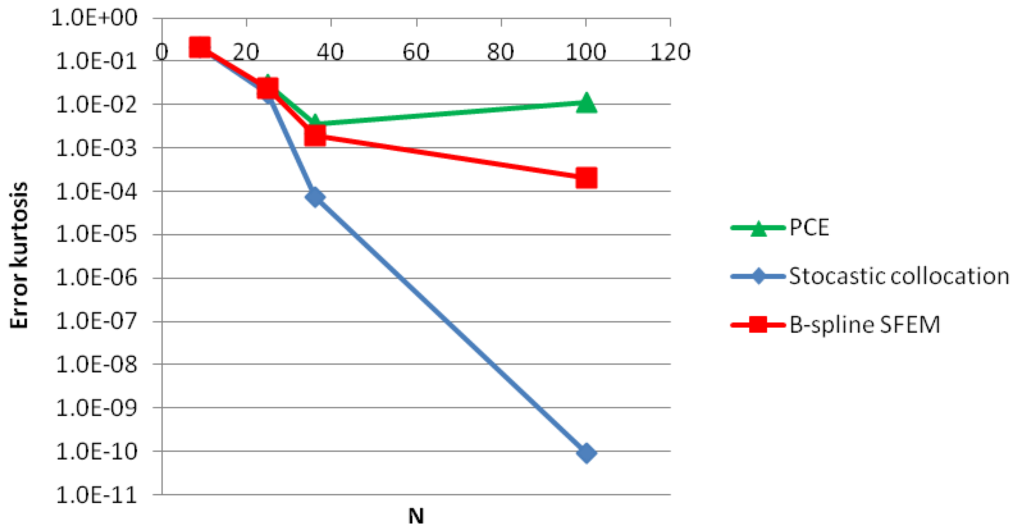

Figure 9. Error on the kurtosis.

The figures show that the error obtained with the B-spline approximation of the response decreases as the number of sampling points increases. If we compare the B-spline approximation with the PCE, it can be observed that the first approach performs better than the second as the number of points increases. However, the Stochastic Collocation method leads to the best approximation of the statistical moments of the response. The significant accuracy obtained in this example is due to the increment of the polynomial degree of the approximation with the number of points. It is expected that also in the case of the B-spline SFEM an increase of the order of the basis functions would lead to a remarkable gain in terms of accuracy.

6. Conclusions

In the present paper, a B-spline-based methodology is proposed for the uncertainty analysis of structural systems characterized by random properties or subject to random actions. The procedure is based on the approximation of the system response by means of B-spline surfaces in the space of independent standard normal random variables. Given a set of samples of the response, the coordinates of the control points are the solution of a least square problem. In the present paper, the degree of the B-spline basis functions is limited to three, as usually done in the majority of the applications of B-spline surfaces.

The proposed methodology is compared with the Polynomial Chaos Expansion and the Stochastic Collocation Method in terms of the accuracy in the estimation of the first four statistical moments of the response. The comparison is performed with respect to the radial displacement of a hollow sphere under internal pressure. In the present example, it has been observed that the B-spline based approximation of the response leads to an accurate estimation of the statistical moments. Moreover, the error decreases remarkably as the number of sampling points is increased.

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